The Order of Operations Redesigned

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Please simplify and answer the following:

- $23 – 2 + 12 – 8 =$
- $4 + 3 \times 7 - (5 + 2) \div 3 =$
- Why do we have the order of operations?
- What teaching supports do you give your students for simplifying algebraic expressions?

We asked the same questions to pre-service teachers at NC State…

- $23 – 2 + 12 – 8 = 25$
  - 25% answered incorrectly
- $4 + 3 \times 7 - (5 + 2) \div 3 = 22\ 2/3$
  - 40% answered incorrectly

We also asked: Is the order of operations easy for you?

  - 72% answered yes

We also asked pre-service teachers at NC State - Why do we have the order of operations?

- Don't Know / No response
- Convention
- Procedural Steps
- Make Simpler

We also asked pre-service teachers at NC State…

- What teaching supports do you give your students for simplifying algebraic expressions?
  - PEMDAS
  - Please Excuse My Dear Aunt Sally

Grant

- NCSU Undergraduate Research Grant
  - Creating an online module for teachers to learn a new way to teach the order of operations with CONCEPTUAL understanding (vs. procedural).
  - We are going to design two models and test both with teachers at different schools. Teachers will try this new teaching model with a student. We will talk to teachers about their experiences with the online module and student to create a final online module to public domain.
- Online Module
  - We have a manuscript under review about our new model with supported research and evidence. Teachers will read this first.
  - Learning Section (What we are going to look at today).
  - Application of Knowledge

Please excuse my dear Aunt Sally from your classroom

- Why?
  - She confuses students about the order to carry out multiplication/division and addition/subtraction.
  - She doesn't speak up and tell us about inverse operations and how to handle them in algebraic expressions.
  - She forgets to mention what happens when we have radicals.

Why do we have the order of operations and how did we get them?

- Some old mathematicians sat around the dinner table and decided to create the order of operations so they would all get the same answer.

WRONG!!!!!!

Why do we have the order of operations and how did we get them?

- Excuses the need for parentheses to direct all operations in algebraic expressions.
- It is one tool, one reliable way to derive the correct simplified value of an expression.
- The order of operations is not random, but a power hierarchy.

Order as a Sequence

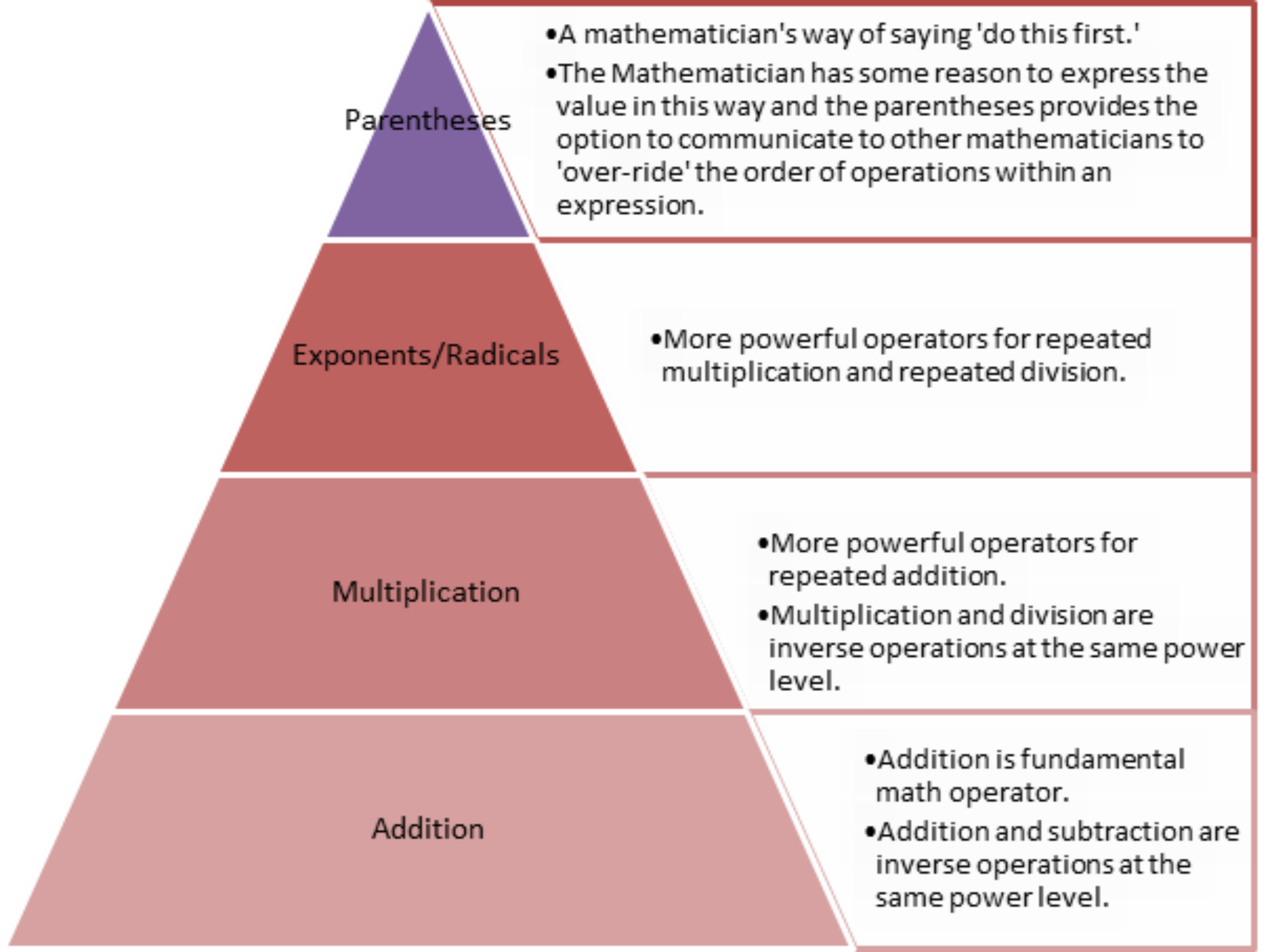
versus

Order as a Rank

+

Let's redefine the "order" of operations

- Order has multiple meanings like "sequence" or "rank"
- Most commonly the order of operations is seen as steps to follow in a particular "sequence"
- We want to start thinking about the order of operations RANKING math operations


Parentheses
• A mathematician's way of saying 'do this first.'
• The Mathematician has some reason to express the value in this way and the parentheses provides the option to communicate to other mathematicians to 'over-ride' the order of operations within an expression.
Exponents/Radicals
• More powerful operators for repeated multiplication and repeated division.
Multiplication
• More powerful operators for repeated addition.
• Multiplication and division are inverse operations at the same power level.
Addition
• Addition is fundamental math operator.
• Addition and subtraction are inverse operations at the same power level.

What do we mean by "ranking" math operations?

- We can RANK math operations by their POWER level
- Addition and subtraction are our fundamentals and building blocks to the other math operators. We rank them at the BOTTOM.

+
The Fundamentals

- Addition is the Fundamental math operator. And it is also the least powerful. It is a basic extension of Counting
- Addition Requires same unit sizes because we are essentially accounting for the amount of like units when we add.

The Fundamentals

- Subtraction is the inverse operator of addition.
  - SAME RANK
- All problems with subtraction can be written in terms of addition.
- Definition of subtraction:
  - $a - b = a + (-b)$

What do we mean by "ranking" math operations?

- Multiplication/division is MORE POWERFUL than addition/subtraction
  - Why? Multiplication is repeated addition. It takes us less work to multiply numbers than to add strings of numbers.

Moving up the power scale

- Multiplication is a MORE POWERFUL math operator for repeated addition. So we move up a rank in the power scale.
  - Example: 3+3+3 = 3 x 3

Moving up the power scale

- Division is the inverse operator of multiplication. SAME RANK
- All problems with division can be written in terms of multiplication.

Multiplication

- More powerful operators for repeated addition.
- Multiplication and division are inverse operations at the same power level.

- Definition of division:
  - $a \div b = a \times \frac{1}{b}$

What do we mean by "ranking" math operations?

- Exponents/radicals are MORE POWERFUL than multiplication/division
  - Why? Exponents are repeated multiplication.

Moving further up the power scale

- Exponents are a MORE POWERFUL math operator for repeated multiplication. So we move up a rank in the power scale again.
  - Example: $3 \times 3 \times 3 = 3^3$

Moving further up the power scale

- Radicals are even more powerful operators for repeated division. Radicals can be seen as fractional exponents.
  - $\sqrt{9} = 9^{\frac{1}{2}} = 3$

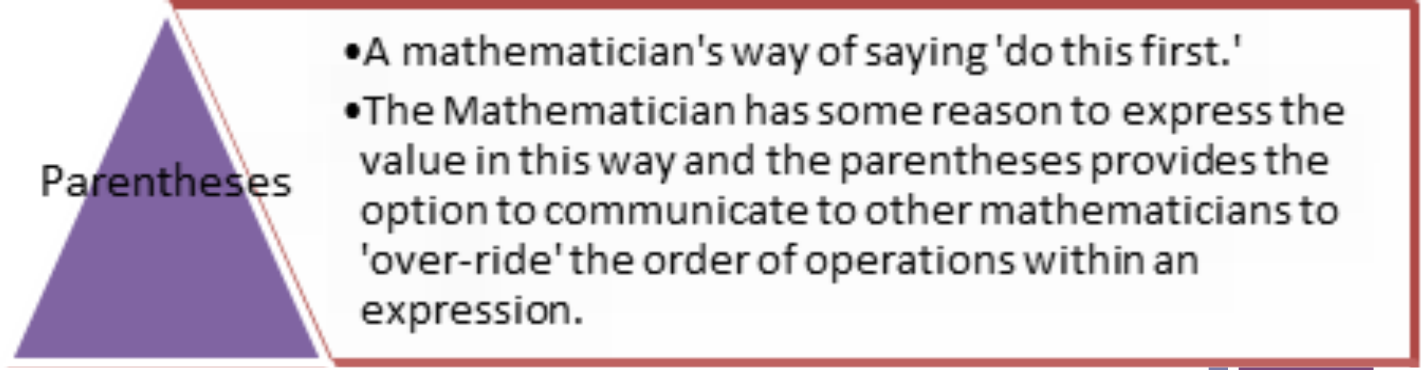
Tip Top

Parentheses

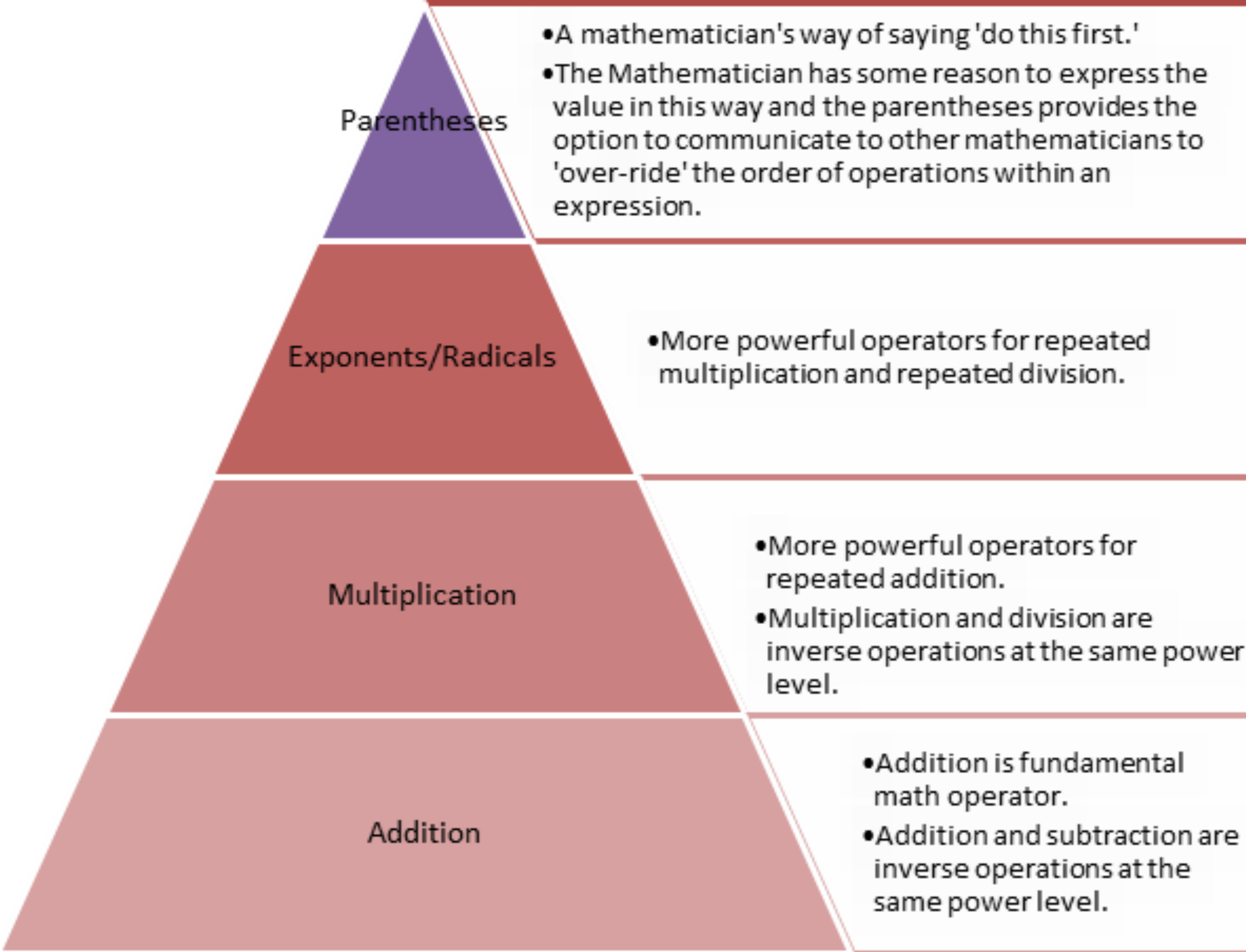
- A mathematician's way of saying 'do this first.'
- The Mathematician has some reason to express the value in this way and the parentheses provides the option to communicate to other mathematicians to 'over-ride' the order of operations within an expression.

- Parentheses are not a math operator. We use them to group other math operators in a way that would not have been understood without parentheses.
- When working inside the set of parentheses we still follow the order of operations.

Tip Top



- All expressions with parentheses can be written without parentheses. We just need to use the distributive property.
  - 9 x (8 + 3)
    - With parentheses first: 9 x (8 + 3)=9 x 11 = 99
    - Rewritten by distributive property:
    - 9 x (8 + 3) = 9 x 8 + 9 x 3 = 72 + 27 = 99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+ The Fundamentals

$$3 - 6 + 9$$

Equal Power, Solve L→R: $\underbrace{3 - 6} + 9 \;=\; -3 + 9 = 6$

Solve with all addition:

$$3 \underbrace{-\,6}_{\text{The definition of subtraction}} + 9 = \;3 + (-6) + 9 \;= -3 + 9 = 6$$

Addition

- Addition is fundamental math operator.
- Addition and subtraction are inverse operations at the same power level.

+ The Fundamentals

Once we have all the same RANK, we use the definition of the inverse so we can solve in any SEQUENCE we want thanks to the Commutative Property

$$3 + (-6) + 9 \quad = \quad (-6) + 9 + 3 \quad = \quad 9 + (-6) + 3$$

Addition

- Addition is fundamental math operator.
- Addition and subtraction are inverse operations at the same power level.

Moving up the power scale

$$3 \div 6 \times 9$$

Equal Power, Solve L→R: $\underbrace{3 \div 6} \times 9 = \frac{1}{2} \times 9 = \frac{9}{2} = 4\frac{1}{2}$

Multiplication

- More powerful operators for repeated addition.
- Multiplication and division are inverse operations at the same power level.

Solve with all multiplication:

$$\underbrace{3 \div 6}_{\text{The definition of division}} \times 9 = 3 \times \frac{1}{6} \times 9 = \frac{1}{2} \times 9 = \frac{9}{2} = 4\frac{1}{2}$$

Moving up the power scale

Once we have all the same RANK, we use the definition of the inverse so we can solve in any SEQUENCE we want thanks to the Commutative Property

$$3 \times \frac{1}{6} \times 9 \quad = \quad \frac{1}{6} \times 3 \times 9 \quad = \quad 9 \times \frac{1}{6} \times 3$$

“The Order of Operations is Wrong”

Three Basic Tools to Attack Expressions

- Using the definition of inverse operations

- Distributing

- Using Power Ranking (order of operations)

Flexibility and Freedom

- Solve $3 \times (4 + 5) - (6 \times 2)^2$ three different ways

Flexibility and Freedom

- Possible Class discussion Example Student Work

- 3 x (10 +2)
- I did it by distributing which led me to 30 + 6 = 36

- 3 x (10 + 2)

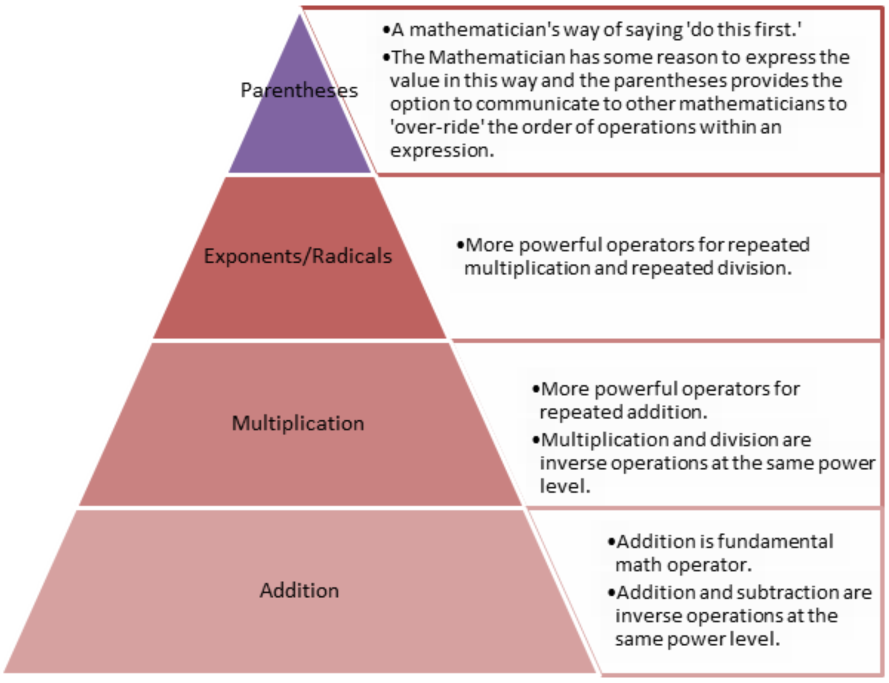
I did it by using the order of operations and simplifying inside the parentheses first then multiplying.

I got 3 x 12 = 36

Summing it all up - Freedom

- The order of operations is ONE way to attack a problem

Summing it all up – Order of Operations is about Rank



1. Parentheses: Apply the order of operations as you work inside.
2. Most powerful: Radicals & Exponents
3. Next powerful: Multiplication & Division
4. Fundamental: Addition & Subtraction

Questions or Comments?

Any feedback for us to improve?

Alternative Pyramid for Understanding

Questions about Math Talk?

- How can I talk to my students about math?